NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

RESEARCH REPORT No. EM-77

Step Discontinuities in Waveguides

W. ELWYN WILLIAMS

NEW YORK UNIVERSITY
COURANT INSTITUTE-LIBRARY
251 Mercer St. New York, N.Y. 10012

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Pl. New York 3, N.Y.

CONTRACT No. AF-19(122)-42

APRIL 1955

NEW YORK UNIVERSITY
Institute of Mathematical Sciences
Division of Electromagnetic Research

Research Report No. EM-77

STEP DISCONTINUITIES IN WAVEGUIDES

W. Elwyn Williams

W. Elwyn Williams

NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.

Morris Kline
Project Director

The research reported in this document has been made possible through support and sponsorship extended by the Air Force Cambridge Research Center, under Contract No. AF-19(122)-42. It is published for technical information and does not necessarily represent recommendations or conclusions of the sponsoring agency.

New York, 1955

Abstract

The method developed by Wiener and Hopf for solving a certain class of integral equations is applied to the problem of a discontinuity in cross-section of a rectangular waveguide. We consider the two-dimensional acoustic problem of two channels of infinite width and different heights joined together with a short step; it is assumed that the fundamental mode is incident in the larger channel traveling towards the step. For values of the channel height such that only the fundemental mode is propagated without attenuation, this problem has been treated in the Wave Guide Handbook; our results are in very good agreement with the results given there. The present analysis is extended to cover the case where the larger channel is capable of supporting the mode above the fundamental without attenuation. Some numerical values are given for the ratio between the reflected and incident energies for various values of the channel and step heights.

Table of Contents

		Page
I.	Introduction	1
II.	Statement of the problem	3
III.	Solution for Ψ	4
IV.	Calculation of reflection coefficient	10
V.	Comparison with the results of the Handbook	13
VI.	The case of $ka/\pi > 1$	15
VII.	Solution of the equations	16
VIII.	Numerical results	23
Appendix A		26
Appendix B		28
References		32

I. Introduction

Problems involving discontinuities in waveguides are of great practical importance and a considerable amount of attention has been devoted to them in recent years. The two types of discontinuities that have been most extensively treated for a rectangular guide are the bifurcation of the guide by a semi-infinite plane and a step discontinuity in the guide cross-section. The problem of the bifurcation may be solved in terms of known functions by applying the well-known Wiener-Hopf technique for solving integral equations. The methods for solving the problem of a step discontinuity which have appeared in the literature are, however, in no way based on the Wiener-Hopf approach. Two distinct methods have been employed; the first consists of expanding the field components on either side of the discontinuity in terms of the appropriate eigenfunctions for the region considered, and then matching the components across the plane of discontinuity. The second method used is the equivalent static approach as described in the Wave Guide Handbook [1]. Since there is obviously a physical relationship between these two problems it is desirable to solve them both by the same method in order that the solution should display clearly the relationship between them, and this is the object of the present work.

We therefore consider the application of the Wiener-Hopf technique to solve the problem of a step discontinuity in a parallel plate region (or, equivalently, the coupling of two parallel plate waveguides of different height). The problem is formulated as an acoustic one, but it could have been equally-well stated in electromagnetic terms by considering the case in which the only non-vanishing component of the magnetic field is parallel to the edges of the step. It is assumed that the fundamental mode is incident from the larger region on the step and, initally, that the dimensions of both regions concerned are taken to be such that only the fundamental mode may propagate without attenuation. This restriction is later removed and

the case of the larger region being capable of sustaining two non-attenuated modes is treated. The advantage of this method of attack is that it affords a clearer physical picture of the structure of the solution than that obtained by the other methods. In particular, it is possible to separate the reflection coefficient for the fundamental mode into two parts, one of which is expressible in terms of known functions and another which may be obtained from the solution of an infinite set of equations. That part of the reflection coefficient which is known exacly is by far the most dominant part of the complete expression, and thus the percentage error in determining an approximate value of the reflection coefficient is much less than the corresponding error made in the solution of the equations. The analytic part of the solution may be obtained by simple physical arguments from the known solution of the bifurcation problem and thus the present form of the solution gives a distinct connection between the two types of problems.

For the case in which only the fundamental mode is propagated without attenuation in any of the regions considered, it is possible to compare the present results with those given in the Wave Guide Handbook which were obtained by the equivalent static method, and the results of both methods show remarkably good agreement. The form of the infinite set of equations obtained here is such that it is possible to estimate the error made in solving only a finite number of them, and a method is indicated by which this error could be estimated in any particular case. The method used here may be easily extended to solve the problem of a bifurcated waveguide when the material composing one of the bifurcated regions differs in dielectric properties from that of the remainder of the waveguide. The approach may also be used to consider the problem of radiation from a dielectric waveguide, and it is hoped that both of these problems will be treated in future papers. The method can,of course, be applied to the general electromagnetic case since it is known that this problem can be

reduced to the solution of two scalar problems [2], but no attempt is made to consider this more general case. The author has been informed by Prof. N. Marcuvitz that Prof. J. Schwinger has considered the application of the Wiener-Hopf technique to problems similar to the one treated here. An account of his work may appear in a forthcoming book by Marcuvitz and Schwinger.

The author would like to acknowledge the permission of Mr. R.A. Brooker to use the digital computer at Manchester University for most of the computations involved in the text, and also the many helpful suggestions given by Mr. D. S. Jones during the period in which the present work was carried out.

II. Statement of the problem

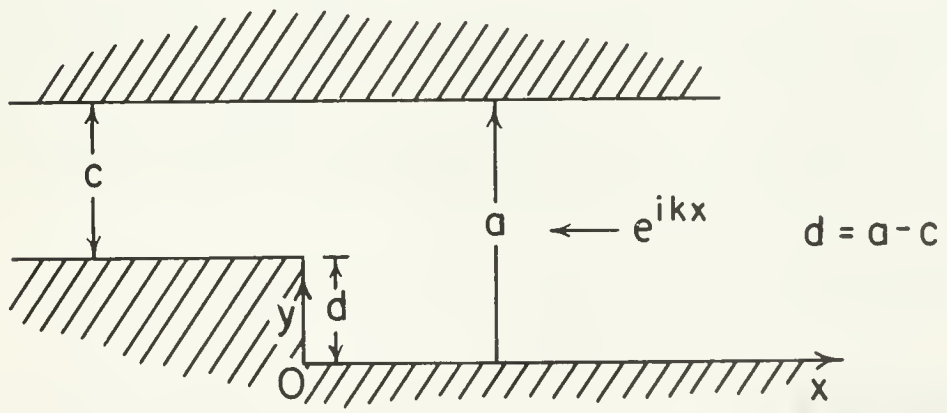


Fig. I

The waveguide is assumed to occupy the region $-\infty \leq x \leq 0$, $d < y < a$; and $0 \leq x \leq \infty$, $0 < y < a$ (see Fig. I), where x, y are Cartesian coordinates with their origin at O. Oz, the third axis of the system, is not shown, and the solution is assumed to be independent of z. A time variation $e^{i\omega t}$ will be

taken as understood throughout the work, and it is assumed that an incident wave e^{ikx} is approaching the step from the right, k being $2\pi/(\text{wavelength})$.

The total velocity potential will be written as $e^{ikx} + \Psi$, and hence a function Ψ is sought which satisfies the following conditions:

(i) $$(\nabla^2 + k^2)\Psi = 0.$$

(ii) $$\frac{\partial \Psi}{\partial x} = -ik \quad \text{on} \quad x = 0, \quad 0 \leq y \leq d$$

(iii) $$\frac{\partial \Psi}{\partial y} = 0 \quad \text{on} \quad y = a, \quad -\infty \leq x \leq \infty,$$
$$\text{on} \quad y = d, \quad -\infty \leq x \leq 0, \text{ and}$$
$$\text{on} \quad y = 0, \quad 0 \leq x \leq \infty \quad .$$

It is convenient to assume that $k = k_r - ik_i$, where $k_i, k_r > 0$; then the form of the radiation condition for Ψ is $\Psi = O(e^{-ik|x|})$ as $|x| \to \infty$. In order to specify the problem uniquely, a condition has to be imposed on Ψ at the edges of the step[3]. This condition is that Ψ is $O(1)$ as the point of observation approaches (0,d) or (0,0), while $|\nabla\Psi|$ is $O(r^{-1/3})$ as the point of observation approaches (0,d) and is $O(1)$ in the neighborhood of the origin -- r being the distance from the point approached. The above conditions now specify the problem completely and uniquely.

III. Solution for Ψ

We shall now define a function $\phi(s,y)$ by

$$\phi(s,y) = \int_{-\infty}^{\infty} e^{-sx}\, \Psi(x,y)\,dx$$

where s is the complex variable $\sigma + i\tau$. From the assumption on Ψ as $|x| \to \infty$, it is clear that the above integral is convergent for $-k_i < \sigma < k_i$. Hence ϕ is a regular function of s in the region $|\sigma| < k_i$. We also define $\phi_-(s,y)$, $\phi_+(s,y)$ as

$$\phi_-(s,y) = \int_{-\infty}^{0} e^{-sx}\Psi(x,y)dx \qquad \phi_+(s,y) = \int_{0}^{\infty} e^{-sx}\Psi(x,y)dx.$$

Clearly, from the behavior of Ψ as $|x| \to \infty$, $\phi_-(s,y)$ is a regular function of s for $\sigma < k_i$, and $\phi_+(s,y)$ is a regular function of s for $\sigma > -k_i$. In the region $0 \leq y \leq d$, $\phi_-(s,y)$ is identically zero and hence we have the following equations. For $y \geq d$

$$\frac{d^2\phi}{dy^2} + K^2\phi = 0. \tag{1}$$

For $0 \leq y \leq d$

$$\frac{d^2\phi_+}{dy^2} + K^2\phi_+ = -ik + sf(y) \tag{2}$$

where $f(y) = \Psi(0,y)$ and $K = (s^2+k^2)^{1/2}$, and where that branch of the root is taken which reduces to k when $s = 0$.

The solution of Eq. (2) which satisfies the boundary condition on $y = 0$ is

$$\begin{aligned}\phi_+(s,y) &= A\cos Ky - \frac{ik}{K}\int_0^y \sin K(y-t)dt + \frac{s}{K}\int_0^y f(t)\sin K(y-t)dt \\ &= A\cos Ky - \frac{ik}{K^2}(1-\cos Ky) + \frac{s}{K}\int_0^y f(t)\sin K(y-t)dt.\end{aligned} \tag{3}$$

where A is a function of s but not of y.

If primes are taken to mean derivatives with respect to y, then differentiation of Eq. (3) and then substitution of $y = d$ in the resulting equation gives

$$\phi_+'(s,d) = -KA\sin Kd - \frac{ik}{K}\sin Kd + s\int_0^d f(t)\cos K(d-t)dt. \tag{4}$$

If the value of A from Eq. (4) is substituted in Eq. (3), then this latter equation may be rewritten as

$$K \sin Kd\, \phi_+(s,y) = -\left\{\phi_+'(s,d) + \frac{ik}{K} \sin Kd - s \int_0^d f(t)\cos K(d-t)\,dt\right\} \cos Ky$$

(5)
$$+\left\{\frac{s}{K}\int_0^y f(t)\sin K(y-t)\,dt - \frac{ik}{K^2}(1-\cos Ky)\right\} K \sin Kd \ .$$

By definition $\phi_+(s,y)$ is regular for $\sigma > k_i$. Hence, since $K \sin Kd$ has zeros in $\sigma > -k_i$, the right-hand side of Eq. (3) must vanish for those zeros of $K \sin Kd$ in $\sigma > -k_i$. These zeros will be at $s = K_m$ $(m = 0,1,2,\ldots)$ where

$$K_m = \left(\frac{m^2\pi^2}{d^2} - k^2\right)^{1/2} ,$$

and is positive when real, and positive imaginary when not real (for k real). It is also easy to show that $R\, K_m \geq k_i$. Thus, in order that the right hand side of Eq. (5) vanishes for $s = K_m$, the following relations must hold

$$\phi_+'(K_m,d) = (-)^m K_m \int_0^d f(t)\cos\frac{m\pi t}{d}\,dt \qquad m \neq 0$$

$$\frac{\phi_+'(ik,d)+ikd}{ik} = \int_0^d f(t)\,dt \ .$$

If the expression for f(t) as a Fourier cosine series as obtained from the above expressionsis substituted in Eq. (5), then application of the well-known results for term by term integration of Fourier series, together with some slight manipulation, yields the following equation for $\phi_+(s,y)$

(7) $$K \sin Kd\,\phi_+(s,y) = -\phi_+'(s,y)\cos Ky + \frac{K \sin Kd}{s+ik} + \frac{2Ks}{d} \sin Kd \sum_{n=o}^{\infty} \varepsilon_m \frac{\cos\frac{m\pi y}{d}}{s^2 - K_m^2}$$

where

$$g_m = \frac{(-)^m \phi_+'(K_m,d)}{K_m} \quad m \neq 0, \qquad g_o = \phi_+' \frac{(ik,d)}{2ik} \quad .$$

We now consider the region $y \geq d$. The solution of Eq. (1) which satisfies the boundary condition on $y = a$ is

$$\phi(s,y) = B \cos K(y-a)$$

where B is a function of s only. Since $\partial\Psi/\partial y$ vanishes on $y = d$ for $-\infty \leq x \leq 0$, $\phi_-'(s,d) = 0$ and hence

$$-B K \sin K(d-a) = \phi_+'(s,d).$$

Thus we obtain the following equation for $\phi(s,y)$

$$\phi(s,y) = - \frac{\cos K(y-a)}{K \sin K(d-a)} \phi_+'(s,d).$$

Hence

$$\phi_+(s,d) + \phi_-(s,d) = - \frac{\cot K(d-a)}{K} \phi_+'(s,d). \tag{8}$$

Substitution for $\phi_+(s,d)$ from Eq. (7) into Eq. (8) gives

$$\phi_-(s,d) = \frac{\sin Ka}{K \sin Kc \sin Kd} \phi_+'(s,d) - \frac{1}{s+ik} - \frac{2s}{d} \sum_{m=o}^{\infty} \frac{(-)^m g_m}{s^2 - K_m^2} . \tag{9}$$

On writing

$$\frac{\sin Ka}{K \sin Kc \sin Kd}$$

as

$$\frac{N_+(s)}{N_-(s)}$$

it is seen that Eq. (9) may be rewritten as

$$N_-(s)\phi_-(s,d) = N_+(s)\phi_+'(s,d) - \frac{N_-(s)}{s+ik} - \frac{2s}{d} N_-(s) \sum_{m=o}^{\infty} \frac{(-)^m g_m}{s^2 - K_m^2} .$$

The formulas for $N_+(s)$, $N_-(s)$ are given in Appendix A but a brief summary of their essential properties will be given here. $N_+(s)$ is regular and non-zero

in the region $\sigma > -k_i$, $N_-(s)$ is regular and non-zero for $\sigma < k_i$. Also, as $|s| \to \infty$, $N_+(s) \sim s^{-1/2}$ and $N_-(s) \sim s^{1/2}$.

It is now convenient to rewrite the above equation for $\phi_-(s,d)$ as follows

$$N_-(s)\phi_-(s,d) + \frac{N_-(s)-N_-(-ik)}{s+ik} + \frac{2s}{d} N_-(s)\sum_{m=o}^{\infty} \frac{(-)^m g_m}{s^2 - K_m^2} - \frac{1}{d}\sum_{m=o}^{\infty} \frac{(-)^m N_-(-K_m) g_m}{(s + K_m)}$$

$$= N_+(s)\phi_+'(s,d) - \frac{N_-(-ik)}{s+ik} - \frac{1}{d}\sum_{m=o}^{\infty} \frac{(-)^m N_-(-K_m) g_m}{(s + K_m)} . \tag{10}$$

Clearly the left hand side of equation (10) is regular for $\sigma < k_i$ while the right hand side is regular for $\sigma > -k_i$; hence both sides are equal to an integral function in the common strip $|\sigma| < k_i$. This integral function may be determined from examination of the growth of both sides of equation (10) as $|s| \to \infty$. In order to do this we require the behavior of $\phi_+'(s,d)$ as $|s| \to \infty$ in $\sigma > -k_i$ and the edge condition shows that $\phi_+'(s,d) \sim s^{-2/3}$ as $|s| \to \infty$. The integral function is now easily seen to be zero and we have

$$N_+(s)\phi_+'(s,d) = \frac{N_-(-ik)}{s+ik} + \frac{1}{d}\sum_{m=o}^{\infty} \frac{(-)^m N_-(-K_m) g_m}{s + K_m} . \tag{11}$$

From Eq. (11), on substituting $s = K_n$, the following set of equations is obtained to determine the $\phi_+'(K_m,d)$

$$N_+(K_n)a_n = \frac{1}{K_n+ik} + \frac{1}{d}\sum_{m=1}^{\infty} \frac{N_-(-K_m)a_m}{K_m(K_m+K_n)} + \frac{a_o N_-(-ik)}{2ikd(K_n+ik)} \quad ,(n = 0,1,2,\cdots) \tag{12}$$

where $a_n = \phi_+'(K_n,d) \,/\, N_-(-ik)$.

For the case in which only one mode is propagated without attentuation in both regions of Fig. I, it is possible to reduce the above set of equations to a real set as follows: we define $L_+(s) = N_+(s)(s+ik)$, $L_-(s) = N_-(s)/(s-ik)$.

From the formulas for $N_+(s)$ and $N_-(s)$ it is clear that $L_+(K_n)$ is real for $n \geq 1$. Hence Eq. (12) may be rewritten as

$$(13)\qquad L_+(K_n)a_n = 1 - \frac{1}{d}\sum_{m=1}^{\infty}\frac{a_m L_-(-K_m)}{K_m(K_m+K_n)}(K_mK_n-k^2) - \frac{ik}{d}\sum_{n=1}^{\infty}\frac{a_m L_-(-K_m)}{K_m} - \frac{a_o L_-(-ik)}{d}\,.$$

From the asymptotic behavior of $\phi_+'(s,d)$ and $N_-(s)$ it is seen that the terms of the second series in Eq. (13) are $O(m^{-13/6})$ as $m \to \infty$, hence this series is convergent.

Putting $n = 0$ in Eq. (13), we obtain

$$(14)\qquad L_+(ik)a_o = 1 - \frac{2ik}{d}\sum_{m=1}^{\infty}\frac{a_m L_-(-K_m)}{K_m} - \frac{a_o L_-(-ik)}{d}\,.$$

If the value for the series

$$\sum_{m=1}^{\infty} a_m L_-(-K_m)\,/\,K_m$$

as obtained from Eq. (14) is substituted into Eq. (13) then the following system of equations is obtained

$$(15)\qquad L_+(K_n) = \frac{1}{2}\left\{1 + a_o L_+(ik) - \frac{a_o}{d}L_-(-ik)\right\} - \frac{1}{d}\sum_{m=1}^{\infty} a_m \frac{L_-(-K_m)(K_mK_n-k^2)}{K_m(K_m+K_n)}$$

$$n = 1,2,\cdots\,.$$

Defining a new set of constants c_n for $n \geq 1$ by

$$c_n = \frac{2a_n}{1 + a_o L_+(ik) - \frac{a_o}{d}L_-(-ik)}$$

it is seen that these will satisfy the following set of equations

$$(16)\qquad L_+(K_n)c_n = 1 - \frac{1}{d}\sum_{m=1}^{\infty}\frac{c_m L_-(-K_m)(K_mK_n-k^2)}{K_m(K_m+K_n)}\,,\qquad (m = 1,2,\cdots).$$

The set of Eq. (16) is real only when the fundamental mode is propagated without attenuation. Finally substitution for a_n into Eq. (14) yields the following equation for a_o.

$$(17) \qquad \frac{L_+(ik)a_o - 1 + (a_o/d)L_-(-ik)}{-L_+(ik)a_o - 1 + (a_o/d)L_-(-ik)} = ig$$

where $$g = \frac{k}{d} \sum_{m=1}^{\infty} \frac{c_m L_-(-K_m)}{K_m} .$$

IV. Calculation of reflection coefficient

In problems involving discontinuities in waveguides, the quantity which is of physical importance is the reflection coefficient. In this section we calculate this for the fundamental mode when $ka/\pi < 1$.

The reflection coefficient is the residue of $\phi_+(s,y)$ at $s = -ik$ and, from Eq. (7), it is given by

$$\frac{\phi_+'(-ik,d)}{2ikd} + 1 + \frac{g_o}{d} .$$

We have from Eq. (11)

$$\phi_+'(-ik,d) = \lim_{s \to -ik} \frac{1}{N_+(s)} \left\{ \frac{N_-(-ik)}{s+ik} + \frac{1}{d} \sum_{m=o}^{\infty} \frac{N_-(-K_m)\phi_+'(K_m,d)}{K_m(s+K_m)} \right\}$$

and

$$N_+(s) \sim \frac{-a}{2ikdc} \frac{N_-(-ik)}{s+ik} \qquad \text{as} \qquad s \to -ik.$$

Thus the reflection coefficient R is given by

$$R = \frac{d}{a} \left(1 + \phi_+' \frac{(ik,d)}{2ikd} \right)$$

Substitution in equation (17) for $\phi_+'(ik,d)$ gives

$$ig = \frac{\frac{aR}{d} + \frac{L_+(ik)}{L_-(-ik)}\left\{\frac{aR}{d} - 1\right\}d}{\frac{aR}{d} - \frac{L_+(ik)}{L_-(-ik)}\left\{\frac{aR}{d} - 1\right\}d} \qquad (18)$$

For simplicity it is now convenient to introduce the non-dimensional expressions $H_+(s)$, $H_-(s)$ defined by

$$H_+(s) = \sqrt{\frac{dc}{a}}\, L_+(s), \qquad H_-(s) = \sqrt{\frac{a}{dc}}\, L_-(s).$$

If this change of notation is made in equation (18), then, writing α for c/a, we obtain

$$\frac{\frac{R}{(1-\alpha)}\left\{1 + \frac{H_+^2(ik)}{\alpha}\right\} - \frac{H_+^2(ik)}{\alpha}}{\frac{R}{(1-\alpha)}\left\{1 - \frac{H_+^2(ik)}{\alpha}\right\} + \frac{H_+^2(ik)}{\alpha}} = ig \qquad (19)$$

Equation (19) may be inverted to give

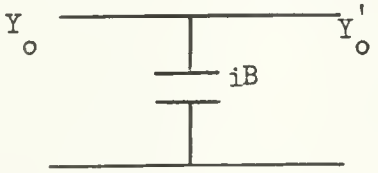
$$R = \frac{\delta(1+ig)H_+^2(ik)}{\alpha+H_+^2(ik)-ig(\alpha-H_+^2(ik))} \qquad (20)$$

where δ is written for $1-\alpha$. The expressions in Appendix A clearly indicate that $|H_+(ik)| = 1$ and hence that $H_+(ik)$ may be written as $e^{i\theta}$ (with θ real).

It is instructive to consider the physical significance of the expression defined by Eq. (20) when $g = 0$. This expression may be derived from physical considerations, as will now be shown. We assume that the step is absent and superimpose a field Ae^{-ikx} incident from the region $0 \leq y \leq d$, $x \leq 0$ on the original field. This incident field will cause a wave AR_1e^{+ikx} to be reflected back in the negative x-direction (R_1 may be calculated by use of the Wiener-Hopf technique) and we thus find that the total velocity component of the potential in the fundamental mode for $y \leq d$, $x \leq 0$ is

$$AR_1 e^{ikx} + e^{ikx} + Ae^{-ikx}$$

and we choose A so that the partial derivative of this expression with respect to x vanishes on the line $x = 0$, $0 \leq y \leq d$. If this is done, then the transmission coefficient for the problem of a wave Ae^{-ikx} incident in the region formerly occupied by the step is seen to agree with the expression of Eq. (20) for $g = 0$. Thus we see that the expression (20) illustrates the connection which was alluded to in Section I between the bifurcation and the step problems. The assumption $g = 0$ of course merely implies that Y is constant over the short vertical part of the step. For $ka < \pi$, it is possible to compare the present results with those given in the handbook, and, since the solution given there is an equivalent circuit formulation, the present results have to be recast in this form. The equivalent circuit is:



$$\frac{Y_o}{Y_o'} = \frac{1}{\alpha}$$

where Y_o, Y_o' are the characteristic admittances of the two guides and B is purely real. $\frac{B}{Y_o'} = C$ may be expressed in terms of the reflection coefficient R, and we have

$$iC = -\frac{\{(\alpha+1)R+(\alpha-1)\}}{\alpha(R-1)} \tag{21}$$

If this expression for R in terms of g is inserted in Eq. (21) then we obtain

$$C = \frac{\delta}{\alpha}\left\{\frac{\sin\theta + g\cos\theta}{\cos\theta - g\sin\theta}\right\}. \tag{22}$$

Inspection of this expression for C shows that it may be obtained by applying the above simple argument for determining the reflection coefficient to the case in which the bifurcation extends a distance $\tan^{-1} g$ to the right of the origin, and we then obtain the exact form of C. Thus the reflection coeffi-

cient for the step problem is the same as the transmission coefficient for the bifurcation problem mentioned above, when the bifurcation extends a distance $\tan^{-1}g$ further towards the positive x-direction than the original step structure.

V. Comparison with the results of the Handbook

Some simplification will be afforded if Eq. (16) is rewritten in dimensionless form by making the substitutions given in Section IV and defining a new set of constants d_n as $c_n d^{-1/2}$. The following equation is then satisfied by the d_n's:

$$H_+(K_n)d_n = \alpha^{\frac{1}{2}} - \alpha \sum_{m=1}^{\infty} \frac{d_m(\mu_m\mu_n-\varepsilon^2)}{\mu_m(\mu_m+\mu_n)} H_-(-K_m) \tag{23}$$

where $\varepsilon = kd/\pi$ and $\mu_m^2 = m^2-\varepsilon^2$. The function g may now be written in dimensionless form as $\varepsilon f \alpha^{1/2}$ where

$$f = \sum_{m=1}^{\infty} d_m H_-(-K_m) \Big/ \mu_m .$$

The formulas are now in the simpliest form possible and are suitable for comparison with the results of the Handbook. First of all we shall consider the special case of neglecting all powers of $\varepsilon \geq 2$. To this order of approximation it may be shown that for $m > 0$

$$H_+(K_m) = \frac{A^m m! \left(m(1-\delta)/\delta\right)!}{(m/\delta)!}$$

where

$$A = \frac{\left(\frac{\delta}{1-\delta}\right)^{\frac{1-\delta}{\delta}}}{\delta^{1/\delta}} .$$

The argument of $H_+(ik)$ may be shown to be $\varepsilon \log A$, and we thus have

$$\frac{C}{\varepsilon} = \frac{\delta}{\alpha}\left(\log A + \alpha^{1/2} f\right).$$

The amount of computation involved in this special case is fairly small,

since to the present order of approximation the μ_m's reduce to m. Eqs. (23) have been solved on the assumption that all the coefficients, apart from the first five, vanished. This gives for $\delta = \frac{1}{2}$, the result C/ε ($\varepsilon = 0$) = 1.556, while the corresponding result in the Handbook is 1.364. For $\delta = .8$ the present method gives $\frac{C}{(ka/\pi)} = 2.45$ while the value in the Handbook is 2.48.

In order to test further the accuracy of the present approach, the general expressions developed in Appendix A were used to calculate $H_-(-\mu_m)$ for certain values of ε, and an approximate solution of Eq. (23) was obtained. The calculations were carried out for $\varepsilon = 4$, $\delta = \frac{1}{2}$; $\varepsilon = .64$, $\delta = .8$; and the values obtained for C were .954 and 3.67. The values given in the Handbook are .96 and 3.73. The values of ε and δ were chosen so that ka/π ranges from 0 tc .3; hence if there were any large deviation between our results and those in the Handbook, it would have been evident in the numerical results. It is thus evident that there is very good agreement between both methods, and that the solution of the first five equations is sufficient to give a very accurate answer. In fact, complete neglect of g produces a percentage error of less than 30 in the expression for C while the percentage error in R will of course be considerably less, (i.e., a very large error in determining g does not produce an equally large error in the expressions for R and C). This indicates that the assumption of a constant value for Ψ on the vertical step is a fairly satisfactory one. It would be possible to obtain an estimate for the error made in the above calculations by using an approach similar to that which will be indicated later for $\varepsilon/\delta > 1$. However, since the above numerical values are sufficient to indicate the accuracy of the method, it does not appear worthwhile to enter into a detailed discussion of estimating the error for this case.

VI. The case of $ka/\pi > 1$

We now consider the modifications which have to be made to the above analysis when $ka/\pi > 1$ but kc/π, $kd/\pi < 1$. One difference is that there will be a term in Ψ which behaves as $\exp(-i\eta x)$, where

$$\eta = (k^2 - \pi^2/a^2)^{1/2},$$

as $x \to \infty$ and η is now real. The coefficient of $\exp(-i\eta x)$ may be found after some manipulation, to be,

$$- \frac{\cos\pi y}{a} \frac{2\sqrt{2}\,\varepsilon\sqrt{\alpha\delta}}{(\beta+\varepsilon)} \left\{ 1 - \frac{2i(\varepsilon-\beta)}{\delta(1+i\varepsilon f\alpha^{1/2})} \sum_{m=1}^{\infty} \frac{e_m(\mu_m+i\varepsilon)}{\mu_m - i\beta} \right\} \mathrm{Re}^{i\phi}$$

where $e_m = d_m H_-(-K_m)\mu_m$, $\beta = (\varepsilon^2-\delta^2)^{1/2}$ and

$$\phi = -\left\{ S_1(\beta,\delta,0) + S_1\left(\frac{(1-\delta)\beta}{\delta},\alpha,0\right) - S_2\left(\frac{\beta}{\delta},1,0\right) - S_1(\varepsilon,0,0) \right.$$
$$\left. - S_1\left(\frac{\varepsilon(1-\delta)}{\delta},0,0\right) + S_2\left(\frac{\varepsilon}{\delta},0,0\right) - \frac{(\varepsilon-\beta)}{\delta}\left(1 + \alpha\log\alpha + \delta\log S\right) \right\}$$

and

$$S_N(x,\alpha,\beta) = \sum_{n=N}^{\infty} \left\{ \sin^{-1}\frac{x}{\sqrt{(n-\beta)^2-\alpha^2}} - \frac{x}{n} \right\}.$$

The S_N's are tabulated in the Handbook.

The above expressions are now in the most convenient dimensionless form for numerical computation.

Another different feature of the case $ka/\pi > 1$ is that the functions $H_-(-K_n)$ are no longer real and we have $H_+(K_m) = \frac{1}{\delta}(\mu_m+i\beta)G_+(K_m)$ where the G_+'s are real. The equation for the new unknowns e_n defined above is

$$\mu_n e_n H_+^2(K_n) = \alpha^{\frac{1}{2}} - \alpha \sum_{m=1}^{\infty} \frac{e_m(\mu_m\mu_n-\varepsilon^2)}{(\mu_m+\mu_n)} \cdot \tag{24}$$

Eq. (24) may be written as two sets of real equations as follows

$$\frac{\mu_n f_n}{\alpha\delta^2} G_+^2(K_n)(\mu_n^2-\beta^2) - 2\beta \frac{\mu_n^2 g_n G_+^2(K_n)}{\alpha\delta^2} = \alpha^{-\frac{1}{2}} - \sum_{m=1}^{\infty} f_m \frac{(\mu_m\mu_n-\varepsilon^2)}{(\mu_m+\mu_n)} \tag{25}$$

$$\frac{G_+^2(K_n)\mu_n}{\alpha\delta^2} (\mu_n^2-\beta^2) g_n + \frac{2\beta\mu_n^2 G_+^2(K_n)}{\alpha\delta^2} f_n = - \sum_{m=1}^{\infty} g_n \frac{(\mu_m\mu_n-\varepsilon^2)}{(\mu_m+\mu_n)} \tag{26}$$

where $e_n = f_n + ig_n$.

It is possible to reduce Eq. (25) and (26) to two sets of linearly independent equations but since, in the present case, they were solved on the digital computer there does not appear to be any advantage gained by their reduction. The above equations also have the advantage that they are in a form from which it is fairly simple to estimate the error involved in their solution.

VII. Solution of the equations

In this section we shall indicate two approaches which may be utilized to estimate the error made in solving only a finite number of Eqs.(25) and (26). It is of course practically impossible to make a precise statement regarding this error, due to the complexity of the equations. Thus some of the following analysis would have to be modified in certain cases but the main scheme of attack will always remain the same. Solving only a finite number of equations is of course equivalent to assuming that the coefficients on the right hand sides of Eqs.(25) and (26) are non-zero only for a finite number of terms. The basis of the approach used here is to estimate the error made on the right hand sides of the equations by making this assumption

and then attempting to obtain bounds for the error made in determining the coefficients by using the known estimate of the error made on the right hand sides by cutting off the series after a finite number of terms.

It is clear that all the terms of the form $(\mu_m\mu_n-\varepsilon^2)$, $(\mu_n^2-\beta^2)$ occurring in Eqs.(25) and (26) will be positive if $\mu_1^2-\varepsilon^2$ is positive and thus a criterion for all these terms to be positive is that $\varepsilon < 1/\sqrt{2}$. This case will be considered first. Examination of the equations clearly indicates that it is not possible to make an exact analytic statement concerning the forms of f_n and g_n and, in particular, it is not possible to assign definite signs to them. It is clear, however, that they will ultimately attain a fixed sign and it does not appear reasonable for them to change sign more than once as n increases. In any case, it is possible to say what combination of signs of f_n and g_n cannot occur.

If it is assumed that the f_n's are all negative, then Eq. (25) shows that the g_n's are all negative and then from Eq. (26) this last statement implies that the f_n's are all positive. Hence the initial assumption that the f_n's are all negative is incorrect since it leads to a contradiction. A similar argument shows that the g_n's may not all be positive. The assumption that the f_n's are positive and such that the right hand side of Eq. (25) is negative leads to the result that the g_n's are all positive and hence a solution with the f_n's satisfying this condition is not permissible.

The predominance of the left hand side of Eqs.(26) and (27) suggests that the form of solution most likely to occur in practice is that in which the f_n's are all positive while the g_n's are all negative. For the case of $\varepsilon > 1/\sqrt{2}$ it is impossible to make any statements whatsoever regarding the signs of f_n and g_n since for the first few values of n the coefficients $(\mu_n^2-\beta^2)$, $(\mu_m\mu_n-\varepsilon^2)$ will be negative and the above argument concerning the admissible combinations of signs will no longer be valid. For the purposes of the subsequent analysis

it will be assumed that the form of the solution is such that all the g_n's are negative while all the f_n's, apart from the first r, are positive. The form of the terms on the left hand side of the equations suggests that the present assumption is the most likely one but it is not the only one which may occur. It has the advantage, however, of being sufficiently simple to illustrate the method that could be employed in any particular example without making the analysis too elaborate and difficult to follow.

The above arguments break down in certain critical cases e.g., when the right hand side of Eq. (25) becomes negative for n greater than a certain value. No criteria can be given for these critical cases and each particular problem has to be investigated separately. However the general approach given here may be modified to suit any particular case under consideration. In some cases the work involved may be considerable. We now proceed with a more detailed study of the equations. Eq. (25) gives that, for $n > r$,

$$(27) \qquad 0 < f_n \frac{\mu_n(\mu_n^2-\beta^2)G_+^2(K_n)}{\alpha\delta^2} < \alpha^{-\frac{1}{2}} - \sum_{m=1}^{r} \lambda_{mn} f_m$$

where $\lambda_{mn} = (\mu_m\mu_n-\varepsilon^2) \Big/ (\mu_m+\mu_n)$.

Defining N as some fixed integer > r we have for $n \geq N$

$$(28) \qquad \frac{1}{\mu_n(\mu_n^2-\beta^2)} < \frac{1}{n^3(1-\varepsilon^2/N^2)^{1/2}(1-(\beta^2+\varepsilon^2)/N^2)} .$$

Hence we have

$$(29) \qquad 0 < f_n < \frac{C}{n^2}\left(\alpha^{-1/2} - \sum_{m=1}^{r} \lambda_{mn} f_m\right)$$

where C is a constant which may be determined from inequality (28) and the inequality B13 for $G_+(K_n)$. Similarly it may be deduced from Eq. (26) that for $n \geq N$

(30) $$0 > g_n > -\frac{D}{n^3}\left(\alpha^{-1/2} - \sum_{m=1}^{r} \lambda_{mn} f_m\right)$$

where

$$D = \frac{2\beta C}{1 - \frac{\beta^2+\varepsilon^2}{N^2}}$$

These inequalities for f_n and g_n may now be used to estimate the series

$$\sum_{N+1}^{\infty} \lambda_{mn} f_m , \qquad \sum_{N+1}^{\infty} \lambda_{mn} g_m .$$

From equation (29) we have

$$\sum_{m=N+1}^{\infty} \lambda_{mn} f_m < A \sum_{m=N+1}^{\infty} \frac{1}{m^2}\left(\alpha^{-1/2} - \sum_{S=1}^{r} \lambda_{Sm} f_S\right)$$

(31) $$< C \int_N^{\infty} \frac{\mu_n x}{(\sqrt{x^2+\varepsilon^2}+\mu_n)} \frac{1}{x^2}\left(\alpha^{-1/2} - \sum_{S=1}^{r} \frac{f_S \mu_S \sqrt{x^2-\varepsilon^2}}{(\sqrt{x^2-\varepsilon^2}+\mu_S)}\right)$$

and the right hand side of this equation may be written as

$$j_n + \sum_{S=1}^{r} f_S \mu_S j_{Sn}$$

where

$$j_n = C\alpha^{-1/2} \frac{\mu_n^2}{n^2} \log\left(\frac{\mu_N+\mu_n}{N}\right) + \frac{\varepsilon^2}{n^2}\left[\frac{\pi}{2} - \tan^{-1}\frac{\mu_N}{\varepsilon}\right]$$

$$j_{Sn} = \frac{\mu_n C}{\mu_n-\mu_S}\left[\frac{\mu_S}{S} \log\left\{\frac{\sqrt{N(\mu_S+N)}+\mu_S\mu_N-\varepsilon^2}{(1+\mu_S)(\mu_S+\mu_N)}\right\} - \frac{\mu_n}{n} \log\left\{\frac{\sqrt{N(\mu_n+N)}+\mu_n\mu_N-\varepsilon^2}{(1+\mu_n)(\mu_n+\mu_N)}\right\}\right] \quad S \neq n$$

$$j_{nn} = -\frac{\varepsilon^2\mu_n}{n^3} C \log \frac{\sqrt{N(\mu_n+N)}+\mu_n\mu_N-\varepsilon^2}{(1+\mu_n)(\mu_n+\mu_N)} + C\frac{\mu_n^2}{n^2}\left[1 - \frac{N}{\mu_N+\mu_n}\right] .$$

Similarly from equation (30)

$$\sum_{N+1}^{\infty} \lambda_{mn} g_m > - \omega_n + \sum_{S=1}^{r} f_S \mu_S \omega_{Sn}$$

where

$$\omega_n = \frac{D\mu_n}{n^2}\left\{1 - \frac{\mu_N}{N}\right\} + \frac{D\mu_n}{n^3} \log \left\{\frac{\varepsilon^2 - \mu_n\mu_N + nN}{(\mu_N+\mu_n)(N-\mu_n)}\right\}$$

$$\omega_{Sn} = \frac{D}{\mu_n - \mu_S}\left[\frac{(n^2-S^2)}{n^2S^2}\left\{\frac{\pi}{2} - \tan^{-1}\frac{\mu_N}{\varepsilon}\right\} + \frac{\mu_S}{S^2}\log\frac{(\mu_S+\mu_N)}{N} - \frac{\mu_n}{n^2}\log\frac{(\mu_n+N)}{N}\right] \quad S \neq n$$

$$\omega_{nn} = \frac{(2\varepsilon^2-n^2)}{n^4}\mu_n D\left\{\frac{\mu_n}{\varepsilon}\left[\frac{\pi}{2} - \tan^{-1}\frac{\mu_N}{\varepsilon}\right] - \log\frac{(\mu_N+\mu_n)}{N}\right\}$$

$$- \frac{D\mu_n^2}{n^2(\mu_N+\mu_n)} + \frac{D\mu_n}{n^2\varepsilon^2}\left[\frac{\pi}{2} - \tan^{-1}\frac{\mu_N}{\varepsilon}\right] .$$

It is now convenient to rearrange the first N equations of each of Eqs. (26) and (27). They may be rewritten as

$$(32) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\left\{\mu_n(\mu_n^2-\beta^2)f_n - 2\beta\mu_n^2 g_n\right\} + \sum_{1}^{N} \lambda_{mn} f_m = \alpha^{-1/2} - \sum_{N+1}^{\infty} \lambda_{mn} f_m$$

$$(33) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\left\{\mu_n(\mu_n^2-\beta^2)g_n + 2\beta\mu_n^2 f_n\right\} + \sum_{1}^{N} \lambda_{mn} g_m = - \sum_{N+1}^{\infty} \lambda_{mn} g_m .$$

If the right hand sides of equations (32) and (33) were known exactly then it would be possible to obtain the exact values of f_n and g_n. Unfortunately this is not the case, but the bounds between which these right hand sides lie are known in terms of $f_1 \cdots f_r$. For equation (32) the bounds are $\alpha^{-1/2}$ and

$$\alpha^{-1/2} - j_n + \sum_{S=1}^{r} f_S \mu_S j_{Sn} ,$$

while the corresponding values for Eq. (33) are 0 and

$$-\omega_n + \sum_{S=1}^{r} f_S \mu_S \omega_{Sn} \,.$$

If f_n^*, g_n^*, $n \leq l \leq N$ are defined as the solutions of Eq. (32) and (33) when the series on the right hand side are neglected, then

$$(34) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\left\{\mu_n(\mu_n^2-\beta^2)(f_n-f_n^*)-2\beta\mu_n^2(g_n-g_n^*)\right\} = -\sum_{N+1}^{\infty}\lambda_{mn}f_m - \sum_{1}^{N}\lambda_{mn}(f_m-f_m^*)$$

$$(35) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\left\{\mu_n(\mu_n^2-\beta^2)(g_n-g_n^*)+2\beta\mu_n^2(f_n-f_n^*)\right\} = -\sum_{N+1}^{\infty}\lambda_{mn}g_m - \sum_{1}^{N}\lambda_{mn}(g_m-g_m^*).$$

If ε_n and ν_n are defined by $|f_n-f_n^*|$, $|g_n-g_n^*|$ then Eq. (34) and (35) yield the following inequalities

$$(36) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\mu_n|\mu_n^2-\beta^2|\varepsilon_n \leq \sum_{N+1}^{\infty}|\lambda_{mn}||f_m| + \sum_{1}^{N}|\lambda_{mn}|\varepsilon_m + \frac{2\beta\mu_n^2}{\alpha\delta^2}G_+^2(K_n)\nu_n$$

$$(37) \qquad \frac{G_+^2(K_n)}{\alpha\delta^2}\mu_n|\mu_n^2-\beta^2|\nu_n \leq \sum_{N+1}^{\infty}|\lambda_{mn}||g_m| + \sum_{1}^{N}|\lambda_{mn}|\nu_m + \frac{2\beta\mu_n^2}{\alpha\delta^2}G_+^2(K_n)\varepsilon_n \,.$$

The maximum values of the first series on the right of each of Eqs.(36) and (37) will be denoted by τ_n and ω_n respectively. Addition of these equations then gives

$$(38) \qquad (\varepsilon_n+\nu_n)\left\{\mu_n|\mu_n^2-\beta^2|-2\mu_n^2\beta\right\}\frac{G_+^2(K_n)}{\alpha\delta^2} \leq \tau_n + \omega_n + \sum_{1}^{N}|\lambda_{mn}|(\varepsilon_m+\nu_m).$$

If the coefficient of $\varepsilon_n + \nu_n$ on the right hand side of Eq. (38) is negative then the inequality is trivial, but since ε_n and ν_n are interchangeable in inequalities (36) and (37), it is still possible to obtain a non-trivial form of Eq. (38). If the coefficient of $(\varepsilon_n + \nu_n)$ is defined as ρ_n then application of Eq. (38) to itself gives

$$(39)\qquad \rho_n(\varepsilon_n + \nu_n) \leq \gamma_n + \omega_n + \sum_{m=1}^{N} \frac{|\lambda_{mn}|}{\rho_m} \left\{ \gamma_m + \omega_m + \sum_{S=1}^{N} |\lambda_{mS}|(\varepsilon_S + \nu_S) \right\}.$$

If

$$\left| \sum_{1}^{N} \frac{|\lambda_{mn}|}{\rho_m} \right| < 1$$

then if the above iterative process is carried out an infinite number of times, a definite bound is obtained for $(\varepsilon_n + \nu_n)$. Subtraction of Eqs. (34) and (35) yields the inequality

$$(40)\qquad \frac{G_+^2(K_n)}{\alpha\delta^2} \mu_n(\mu_n^2-\beta^2)(\varepsilon_n-\nu_n) < \gamma_n + \omega_n + \sum_{m=1}^{N} |\lambda_{mn}|(\varepsilon_m+\nu_m) + \frac{2\beta\mu_n^2}{\alpha\delta^2} G_+^2(K_n)(\varepsilon_n + \nu_n).$$

This inequality will be trivial if it yields a result of the form $(\varepsilon_n-\nu_n) < (1+\theta)(\varepsilon_n+\nu_n)$ where $\theta > 0$. However, since addition of Eqs. (34) and (35) produces the above inequality with the coefficients of $\varepsilon_n-\nu_n$, $\varepsilon_n+\nu_n$ reversed, it is always possible to obtain bounds for $\varepsilon_n-\nu_n$, $\varepsilon_n+\nu_n$ and hence for ε_n and ν_n. If $r \neq 0$ then the bounds for ε_n, ν_n will be obtained in terms of $f_1 \ldots f_r$ and hence equations of the form

$$\varepsilon_n \leq \chi_n + \sum_{S=1}^{r} \theta_{nS} f_S$$

are obtained. This new set of equations may again be treated iteratively. It should be noted that the above approach is considerably simplified if $ka/\pi < 1$; in this case there is only one set of equations and its form indicates that the unknowns are positive.

The type of approach indicated above may, with suitable modifications, be applied to estimate the error in any particular case but, since some of the inequalities obtained are rather crude, the following method, which is numerically more complicated, may be employed to give a better estimate of the error.

The system of Eqs. (32) and (33) may be written in matrix notation as

$$A\,\tilde{x} = \tilde{y}$$

where x and y are respectively the transposes of the row vectors $(f_1 \ldots f_N, g_1 \ldots g_N)$, and

$$(a^{-\frac{1}{2}} - \sum_{N+1}^{\infty} \lambda_{m1} f_m, \ldots a^{-\frac{1}{2}} - \sum_{N+1}^{\infty} \lambda_{mN}, f_m, \sum_{N+1}^{\infty} \lambda_{m1} g_m, \ldots \sum_{N+1}^{\infty} \lambda_{mN} g_m)$$

and A is the matrix of the coefficients on the left hand sides of Eqs. (32), (33). If the elements of the inverse of A are denoted by a_{ij} and if the suffix i denotes the i-th component of either column vector, then

$$x_i = \sum_{j=1}^{2N} a_{ij} y_j. \tag{41}$$

Though the exact values of y_i are not known, their upper and lower bounds are given in terms of $f_1 \ldots f_r$ and the upper and lower bounds of the x_i may be obtained as follows.

If, in Eq. (41), the maximum values of y_j are inserted for a_{ij} negative and the minimum values for a_{ij} positive, then the resulting expression gives a lower bound for the x_i's. Similarly an upper bound can be obtained. Hence inequalities of the form

$$\sum_{j=1}^{r} b_{ij} f_j + c_i \leq x_i \leq d_i + \sum_{j=1}^{r} e_{ij} f_j$$

are obtained where the b's, c's, d's and e's are all known in terms of $f_1 \ldots f_r$. Either an iterative or some similar process may now be used to determine upper and lower bounds for $f_1 \ldots f_r$. These bounds may then be used to obtain bounds for the remaining f's and g's.

VIII. Numerical results

The set of equations occurring for the case $ka/\pi > 1$ was solved by assuming that the integer N, referred to in the previous section, was equal to 5. Since there is no particular physical significance in the actual numerical values of the coefficients f_n and g_n, they have not been tabulated. One quantity which

is of particular interest is the reflected energy crossing a plane normal to the x axis which will be proportional to

$$\lim_{x\to\infty} \int_0^a \Psi^* \frac{\partial \Psi}{\partial x}\, dy,$$

where the asterik indicates the complex conjugate. If the coefficient of $e^{-i\eta x}$, referred to in Section VI, is written as $R_1 \cos \pi y/a$, then the above integral may be evaluated to give

$$-ia \left\{ k|R|^2 + \frac{1}{2}\ \eta\ |R_1|^2 \right\} \quad .$$

Hence the (reflected energy/incident energy) ratio is

$$|R^2| \ + \frac{\beta}{2\varepsilon}\ |R_1|^2.$$

The two quantities $|R|^2$, $|R|^2 + \frac{\beta}{2\varepsilon} |R_1|^2 = |R_2|^2$, are given in Table I. The numerical values show that the ratio of the reflected energy to the incident energy increases, for constant step height, as the waveguide height decreases. This, of course, was to be expected on physical grounds. The percentage of the reflected energy which is due to the $|R_1|^2$ tends to increase as the waveguide height decreases for constant step height. This percentage is, however, fairly constant ranging between 19% and 25% for all the values given in the table.

The values obtained for $|R|^2$ were compared with those obtained by assuming that all the coefficients of the set of equations vanish, and the two results were found to differ by only a few percent in all cases. The difference between the two values obtained for $|R_1|^2$ was greater, but this contributes only a small amount to the total energy. It thus appears that a very good estimate may be obtained for the reflected energy by assuming a constant value of the velocity potential on the step.

The general methods of the previous section were not used to estimate the

errors in all the sets of equations concerned. However, in all cases the values obtained for f_1 and g_1 did not differ by more than 5% from those obtained by assuming that f_1 and g_1 were the only non-vanishing coefficients. For the case of $\varepsilon = .9$ the general methods were used and the results were shown to be correct to within about 5%.

	$\varepsilon = .9$			$\varepsilon = .8$				$\varepsilon = .7$				$\varepsilon = .6$	
ka/π	1.8	1.6	1.4	1.7	1.6	1.4	1.2	1.6	1.4	1.2	1.1	1.5	1.4
$\|R\|^2$	.25	.32	.41	.23	.26	.34	.47	.20	.28	.39	.49	.17	.20
$\|R_2\|^2$	.31	.40	.52	.29	.33	.44	.58	.25	.35	.31	.63	.21	.26

Appendix A

In this appendix formulas are developed which may be used for computing the functions $N_+(K_m)$. If the same method is employed as that used by Carlson and Heins [4] then it is fairly easily shown that

$$N_+(s) = \frac{1}{(s+ik)}\sqrt{\frac{a}{dc}}\,\frac{P(\delta)}{P(\frac{\delta}{1-\delta})P(1)}\exp\frac{s\{a\log a - c\log c - d\log d\}}{\pi}$$

$N_-(-s)N_+(s) = -1$ where

$$P(\delta) = \prod_{n=1}^{\infty}\left\{\left(1 - \frac{\varepsilon^2}{\delta^2 n^2}\right)^{\frac{1}{2}} + \frac{s\varepsilon}{k\delta n}\right\}\exp\left[-\frac{s\varepsilon}{k\delta n}\right].$$

It is not really necessary to retain the exponentials in the above infinite products when the complete expression for $N_+(s)$ is being considered, but they must be retained for evaluating each separate product. For generality we shall consider the evaluation of each of the products separately since products of a similar nature occur in the other problems of diffraction theory and only slight modification of the following analysis will be required to cover other cases.

It is clear from the expressions occurring in the text that we must compute expressions of the form

$$\prod_{n=1}^{\infty}\left\{\left(1 - \frac{k^2a^2}{n^2\pi^2}\right)^{\frac{1}{2}} + \frac{K_{ma}}{n\pi}\right\}\exp\left[-\frac{K_{ma}}{n\pi}\right].$$

Though most of the computations involved in this work were carried out on a digital computer it is also possible to obtain fairly convergent expansions in powers of ε^2 for the above products. These expansions are convergent for all the powers of ε and δ that occur and have the advantage of being very much more suitable for numerical computation than straightforward multiplication of the terms of the product. In order to obtain the numerical values for a large

number of values of ε and δ, the digital computer has a distinct advantage over any other form of computer. The program developed for the present problem has also been used for computing infinite products for which the approach used here may not be used (in these products the $n\pi$ in the infinite product was replaced by the n-th zero of a Bessel function or of a transcendental equation involving Bessel and Hankel functions). The following formulas are suitable when not too many values of ε and δ are required, and the computations for $ka/\pi < 1$ were carried out using these formulas. Expansion of

$$\left(1 - \frac{k^2a^2}{n^2\pi^2}\right)^{1/2}$$

and K_m in powers of ε^2, together with some algebraic manipulation, gives the following result for $m \neq 0$

$$\prod_{n=1}^{\infty}\left\{\left(1 - \frac{k^2a^2}{n^2\pi^2}\right)^{\frac{1}{2}} + \frac{K_{ma}}{n\pi}\right\}\exp\left[-\frac{K_{ma}}{n\pi}\right] =$$

$$\prod_{n=1}^{\infty}\left(1 + \frac{m}{\delta n}\right)\exp\left[-\frac{m}{\delta n}\right]\exp P_m(\delta) = e^{-(jm/\delta)}\left\{\frac{m!}{\delta}\right\}^{-1}\exp P_m(\delta)$$

where x! is the factorial function, j is Euler's constant and

$$P_m(\delta) = a_1\varepsilon^4 + a_2\varepsilon^6 + a_3\varepsilon^8 + a_4\varepsilon^{10} + a_5\varepsilon^{12}$$

where

$$8m\delta^3 a_1 = -\zeta(3),\quad 48m^3\delta^5 a_2 = -3m^2\zeta(5) - 2\delta^2\zeta(3)$$

$$2^7m^5\delta^7 a_3 = -5m^4\zeta(7) - 3m^2\delta^2\zeta(5) - 3\delta^4\zeta(3)$$

$$5x2^9m^7\delta^9 a_4 = -35m^6\zeta(9) - 140m^4\delta^2\zeta(7) + 36m^2\delta^4\zeta(5) - 20\delta^6\zeta(3)$$

$$3x2^{10}m^9\delta^{11}a_3 = -63m^8\zeta(11) - 35m^6\delta^2\zeta(9) - 31m^4\delta^4\zeta(7) - 30m^2\delta^6\zeta(5) - 35\delta^8\zeta(3)$$

where $\zeta(s)$ is the Riemann Zeta function i.e.,

$$\zeta(s) = \sum_{n=1}^{\infty}\frac{1}{n^s}\ .$$

The other two products in $N_+(K_m)$ are of the same form as the above with a replaced by c and d respectively and hence the above formulas can be extended to cover the other two products by replacing δ in the expressions for $a_1 \cdots a_5$ by $\frac{\delta}{1-\delta}$, and 1 respectively. These results are valid for $\frac{\varepsilon}{\delta} < 1$ but may be extended to include the case of $1 < \frac{\varepsilon}{\delta} \leq 2$, $\varepsilon < 1$ as follows.

For $\frac{\varepsilon}{\delta} > 1$ the first term in the above infinite product will be complex and hence if we consider the integer n as now ranging from 2 to infinity this new product may be treated exactly as before. The $\zeta(s)$ in the formulas for $a_1 \cdots a_5$ should now be replaced by $\zeta(s)-1$.

The case of $m = 0$ requires special consideration. In this case the products occurring in $N_+(ik)$ may be written in the form

$$\exp i \left\{ \sum_1^{\infty} \left(\sin^{-1} \frac{ka}{n\pi} - \frac{ka}{n\pi} \right) \right\} .$$

The functions occurring in this exponent have been extensively tabulated in the Handbook.

Appendix B

In this appendix we obtain some inequalities for the $G_+(K_m)$ of Section VI. $G_+(K_m)$ is given by

$$G_+(K_m) = \frac{Q_m^{(2)}(\delta)}{Q_m^{(1)}(1) Q_m^{(1)}\left(\frac{\delta}{1-\delta}\right)} \exp \left\{ \frac{1-\delta}{\delta} \log \frac{\delta}{1-\delta} + \frac{1}{\delta} \log \frac{1}{\delta} - \frac{1}{\delta} \right\} \mu_m$$

where

$$Q_m^{(r)}(\delta) = \prod_{n=r}^{\infty} \left\{ \left(1 - \frac{\varepsilon^2}{\delta^2 n^2} \right)^{\frac{1}{2}} + \frac{\mu_m}{\delta n} \right\} \exp \left[- \frac{\mu_m}{\delta n} \right] .$$

Each of the above infinite products satisfies the following type of inequality

$$\prod_{n=s}^{\infty}\left\{1-\frac{\varepsilon^2}{\delta^2 n^2}+\frac{\mu_m}{\delta n}\right\}\exp\left[-\frac{\mu_m}{\delta n}\right] < \prod_{n=s}^{\infty}\left\{\left(1-\frac{\varepsilon^2}{\delta^2 n^2}\right)^{\frac{1}{2}}+\frac{\mu_m}{\delta n}\right\}\exp\left[-\frac{\mu_m}{\delta n}\right]$$

(B2)
$$< \prod_{n=s}^{\infty}\left\{1+\frac{\mu_m}{\delta n}\right\}\exp\left[-\frac{\mu_m}{\delta n}\right]$$

and hence $G_+(K_m)$ satisfies the following inequality

$$\frac{M_m^{(2)}(\delta)(\mu_m)!\left(\frac{\mu_m(1-\delta)}{\delta}\right)!}{\left(\frac{\mu_m}{\delta}\right)!\left(1+\frac{\mu_m}{\delta}\right)}\exp\mu_m\left\{\frac{1-\delta}{\delta}\log\frac{\delta}{1-\delta}+\frac{1}{\delta}\log\frac{1}{\delta}\right\} < G_+(K_m)$$

(B3)
$$\frac{\mu_m!\left(\frac{\mu_m(1-\delta)}{\delta}\right)!}{\left(\frac{\mu_m}{\delta}\right)!\left(1+\frac{\mu_m}{\delta}\right)}\quad\frac{\exp\mu_m\left\{\frac{1-\delta}{\delta}\log\frac{\delta}{1-\delta}+\frac{1}{\delta}\log\frac{1}{\delta}\right\}}{M_m^{(1)}(1)\;M_m^{(1)}\left(\frac{\delta}{1-\delta}\right)},$$

where

$$M_m^{(r)}(\delta) = \prod_{n=r}^{\infty}\left(1-\frac{\varepsilon^2}{\delta n(\mu_m+\delta n)}\right).$$

In order to discuss Eq. (B3) it is necessary to establish inequalities for the factorial functions and the infinite products which occur therein. We shall first of all consider the factorial functions, and, for brevity, shall define a new function $f(\mu_m,\delta)$ by

(B4)
$$f(\mu_m,\delta) = \frac{\mu_m!\left(\mu_m(1-\delta)/\delta\right)!}{\left(\frac{\mu_m}{\delta}\right)!}\exp\mu_m\left\{\frac{1-\delta}{\delta}\log\frac{\delta}{1-\delta}+\frac{1}{\delta}\log\frac{1}{\delta}\right\}.$$

If we apply Binet's second formula [5] for the logarithm of the factorial function to the factorials of Eq. (B4) then $f(\mu_m,\delta)$ may be rewritten as

(B5)
$$f(\mu_m,\delta) = \exp\left\{\frac{1}{2}\log 2\pi\mu_m(1-\delta)+j(m)-1+\left(\frac{\mu_m(1-\delta)}{\delta}+\frac{1}{2}\right)\log\left(1+\frac{\delta}{\mu_m(1-\delta)}\right)\right.$$
$$\left.-\left(\frac{\mu_m}{\delta}+\frac{1}{2}\right)\log\left(1+\frac{\delta}{\mu_m}\right)+\left(\mu_m+\frac{1}{2}\right)\log\left(1+\frac{1}{\mu_m}\right)\right\} = \exp h(m,\delta)$$

where

$$j(m) = \chi(\mu_m+1) + \chi\left(\frac{\mu_m(1-\delta)}{\delta}\right) - \chi\left(\frac{\mu_m}{\delta}+1\right)$$

and $$0 < \frac{B_1}{2x} - \frac{B_2}{12x^3} < \chi(x) < \frac{B_1}{2x} \qquad \text{for } x > 0$$

$B_1 = \frac{1}{6}$, $B_2 = \frac{1}{30}$ are Bernoulli's numbers.

If the inequality

$$x - \frac{x^2}{2} < \log(1+x) < x - \frac{x^2}{2} + \frac{x^3}{3}$$

is applied to the logarithmic terms of B5 then the following inequality may be established for $h(m,\delta)$

$$-\frac{1}{\mu_m^2}\left\{\frac{\delta^2}{4(1-\delta)^2} + \frac{\delta^2}{12} + \frac{1}{4}\right\} - \frac{\delta^3}{6\mu_m^3} < h(m,\delta) - \frac{1}{2}\log 2\pi\mu_m(1-\delta) - j(m)$$

(B6)
$$< \frac{1}{\mu_m^2}\left\{\frac{\delta^2}{12(1-\delta)^2} + \frac{\delta^2}{4} + \frac{1}{12}\right\} + \frac{1}{6\mu_m^3}\left\{\frac{\delta^3}{(1-\delta)^3} + 1\right\} .$$

Similarly, application of the inequality for $\chi(x)$ gives the following inequality for $j(m)$.

$$\frac{B_1}{2\mu_m}\left(\frac{\delta}{1-\delta} + (1-\delta)\right) - \frac{B_1}{12\mu_m^2}\left(\frac{\delta^2}{(1-\delta)^2} + 1\right) - \frac{B_2}{12\mu_m^3}\left(1 + \frac{\delta^3}{(1-\delta)^3}\right) < j(m) < \frac{B_1}{2\mu_m}\left(\frac{\delta}{1-\delta} + (1-\delta)\right)$$

(B7)
$$+ \frac{B_1\delta^2}{2\mu_m^2} + \frac{B_2\delta^3}{12\mu_m^3} + \frac{B_2\delta^4}{4\mu_m^4} .$$

A combination of Eqs. (B6), (B7) now gives the following inequality for $h(m,\delta)$

(B8)
$$\frac{1}{2}\log 2\pi\mu_m(1-\delta) + h^-(m,\delta) < h(m,\delta) < h^+(m,\delta) + \frac{1}{2}\log 2\pi\mu_m(1-\delta)$$

where $h^-(m,\delta)$ is obtained by addition of the left-hand sides of Eqs. (B6), (B7) and $h^+(m,\delta)$ is given by the sum of the right-hand sides of these equations.

In order to complete the discussion of equation B3 it is necessary to establish a lower bound for the infinite products occurring there. They are all of the form

$$(B9) \qquad \prod_{n=s}^{\infty}\left(1-\frac{\varepsilon^2}{\delta n(\mu_m+\delta n)}\right)=\exp\sum_{n=s}^{\infty}\log\left(1-\frac{\varepsilon^2}{\delta n(\mu_m+\delta n)}\right).$$

The following inequality holds for the series of B9

$$\left|\sum_{n=s}^{\infty}\log\left(1-\frac{\varepsilon^2}{\delta n(\mu_m+\delta n)}\right)\right|<\left|\log\left(1-\frac{\varepsilon^2}{\delta s(\mu_m+\delta s)}\right)\right| + \left|+\frac{1}{\delta}\int_s^{\infty}\left|\log\left(1-\frac{\varepsilon^2}{x(\mu_m+x)}\right)\right|dx.\right.$$

The values of s that occur are either 1 or 2; if s = 2, then the above inequality is valid if s is replaced by unity in the integral and if the first term on the right hand side is omitted. We thus require an inequality for

$$\int_{\infty}^{\delta}\log\left(1-\frac{\varepsilon^2}{2x(\mu_m+x)}\right)dx\ .$$

It is easily verified that, after a little manipulation, the following result is valid

$$(B11) \qquad \int_{\infty}^{\delta}\log\left(1-\frac{\varepsilon^2}{2x(\mu_m+x)}\right)dx<\frac{\varepsilon^2}{m}\log\left\{\frac{m+\delta+3\frac{\varepsilon^2}{4m}}{\delta-5\frac{\varepsilon^2}{4m}}\right\}\ .$$

The right hand side of this inequality will be denoted by $\ell(m,\delta)$. Finally, combination of B8 and B11 gives

$$(1-\frac{\delta}{\mu_m})\frac{\delta\sqrt{2\pi(1-\delta)}}{\mu_m^{1/2}}\exp\left\{h^-(m,\delta)-\frac{1}{\delta}\ell(m,\delta)\right\}<G_+(\mu_m)$$

$$(B12) \qquad <\frac{\delta\sqrt{2\pi(1-\delta)}}{\mu_m^{1/2}}\exp\left\{h^+(m,\delta)+\ell(m,1)+\frac{(\delta-1)}{\delta}\ell(m,\frac{\delta}{\delta-1})-\log(1-\frac{\varepsilon^2}{(\mu_m+1)})\right.$$
$$\left.-\log\left(1-\frac{\varepsilon^2(1-\delta)}{\delta(\mu_m+\frac{\delta}{1-\delta})}\right)\right\}$$

We shall only require these inequalities for $m \geq N$ and it is thus possible to obtain simpler forms for this inequality. In the exponent on the right-hand side of B12 all the terms are positive and hence the inequality is still valid if all the terms in the exponent are replaced by their values for m = N; also

μ_m may be replaced by $m(1 - \frac{\varepsilon^2}{N^2})^{1/2}$. In the exponent on the left hand side of B12 all the negative terms may be replaced by their values at m = N, while the positive terms may be replaced by zero. μ_m may now be replaced by m. Thus it is always possible to establish a fairly simple inequality of the form

(B13) $$\frac{A}{m^{1/2}} < G_+(\mu_m) < \frac{B}{m^{1/2}} \qquad m \geq N$$

where A and B are constants independent of m.

References

1. Marcuvitz, N. - Wave Guide Handbook; Radiation Laboratory, 1951.
2. Lewin, L. - Advanced Theory of Wave Guides; Iliffe and Sons, London, 1951.
3. Jones, D.S. - A note on diffraction by edges; Quart.J.Math. and Mech., 3, 189 (1950).
4. Carlson, F. and Heins, A.E. - Diffraction by a pair of parallel plates; Quart. Appl. Math. 4, 313 (1947).
5. Whittaker, E.T. and Watson, G.N. - Modern Analysis; Cambridge University Press, 1940, Chap. 12.

NYU
EM
77 Williams
AUTHOR
Step discontinuities in
TITLE
waveguides.
c.1
C.1
NEW YORK UNIVERSITY
INSTITUTE OF MATHEMATICAL SCIENCES
LIBRARY
25 Waverly Place, New York 3, N. Y.
NEW YORK UNIVERSITY
COURANT INSTITUTE-LIBRARY
251 Mercer Street, New York, N.Y. 10012
DATE DUE
ILL T.J.46. Univ of Missouri
8001 Natural Bridge ...
Missouri 63121
sent 5-9-74
due 5-30-74
GAYLORD